

Remarks on the extended Brauer quotient

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Abstract

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1. Introduction

Let G be a group, and let A be a G -algebra over a complete discrete valuation ring \mathcal{O} with residue field k of characteristic $p > 0$. The well-known Brauer quotient $A(P)$ with respect to a p -subgroup P of G (introduced by M. Broué and L. Puig, see [8, §11]) is an $N_G(P)$ -algebra. If moreover, A is G -interior (that is, A is endowed with a unitary algebra homomorphism $\mathcal{O}G \rightarrow A$), then $A(P)$ becomes a $C_G(P)$ -interior $N_G(P)$ -algebra. This means that one may construct, as in [5, Chapter 9], the $N_G(P)/C_G(P)$ -graded $N_G(P)$ -interior algebra $A(P) \otimes_{C_G(P)} N_G(P)$, so $A(P)$ is extended by automorphisms of P given by conjugation with elements of G .

L. Puig and Y. Zhou [6] extended $A(P)$ by all automorphisms of P , obtaining the so called *extended Brauer quotient* $\bar{N}_A^{\text{Aut}(P)}(P)$ as an $N_G(P)$ -interior k -algebra. The interiority assumption is necessary, because the main feature used is the $\mathcal{O}(P \times P)$ -module structure of A . This construction was further generalized by T. Coconeț and C.-C. Todea [3] to the case of H -interior G -algebras, where H is a normal subgroup of G .

Our aim here is to unify and generalize these constructions, by introducing an extended Brauer quotient of a group graded algebra. The main ingredients of our construction are

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a \bar{G} -graded algebra A , a group homomorphism $P \rightarrow \bar{G}$ (which induces a \bar{G} -grading on the group algebra $\mathcal{O}P$), and a homomorphism $\mathcal{O}P \rightarrow A$ of \bar{G} -graded algebras.

In Section 2 below we recall the Puig and Zhou definition of $\bar{N}_A^{\text{Aut}(P)}(P)$, pointing out its $\text{Aut}(P)$ -graded algebra structure. Our alternative construction in Section 3 is based on the easy observation that if A is a \bar{G} -graded P -algebra with identity component B such that the action of P on \bar{G} is trivial, then the Brauer quotient $A(P)$ inherits the \bar{G} -grading such that the identity component of $A(P)$ is $B(P)$. Here we apply the classical Brauer quotient to the $\text{Aut}(P)$ -graded algebra $\tilde{A} = A \otimes_{\mathcal{O}} \mathcal{O} \text{Aut}(P)$, and we get that $\tilde{A}(P)$ is isomorphic to $\bar{N}_A^{\text{Aut}(P)}(P)$ as $\text{Aut}(P)$ -graded algebras. In Section 4 we construct the extended Brauer quotient of a \bar{G} -graded P -interior algebra A as mentioned above, this time with P acting nontrivially on \bar{G} . We also discuss the exact relationship to the construction from [3]. Section 5 investigates the extended Brauer quotient of tensor products of P -interior algebras, in Section 6 we give an application towards correspondences for covering blocks.

Our general notations and assumptions are standard, and closely follow [8], [5] and [4].

2. The extended Brauer quotient

2.1. The construction of Puig and Zhou

We begin with a p -group P and a P -interior algebra A . Let $\varphi \in \text{Aut}(P)$, and as in [6], we consider the φ -twisted diagonal

$$\Delta_{\varphi}(P) = \{(u, \varphi(u)) \mid u \in P\}.$$

Then the set of $\Delta_{\varphi}(P)$ -fixed elements, is the following \mathcal{O} -submodule of A :

$$A^{\Delta_{\varphi}(P)} = \{a \in A \mid ua = a\varphi(u) \text{ for any } u \in P\}.$$

Further, we consider $Q < P$ and denote by $A_{\Delta_{\varphi}(Q)}^{\Delta_{\varphi}(P)}$ the \mathcal{O} -module consisting of elements of the form

$$\text{Tr}_{\Delta_{\varphi}(Q)}^{\Delta_{\varphi}(P)}(c) = \sum_{u \in [P/Q]} u^{-1} c \varphi(u),$$

where $c \in A^{\Delta_{\varphi}(Q)}$. At last, we denote by $A(\Delta_{\varphi}(P))$ the quotient

$$A(\Delta_{\varphi}(P)) = A^{\Delta_{\varphi}(P)} / \sum_{Q < P} A_{\Delta_{\varphi}(Q)}^{\Delta_{\varphi}(P)},$$

and we obtain the usual Brauer homomorphism

$$\text{Br}_{\Delta_\varphi(P)} : A^{\Delta_\varphi(P)} \rightarrow A(\Delta_\varphi(P)).$$

If K is a subgroup of $\text{Aut}(P)$, it is easily checked that the external direct sum $\bigoplus_{\varphi \in K} A^{\Delta_\varphi(P)}$ is an algebra, while its subset $\bigoplus_{\varphi \in K} \sum_{Q < P} A^{\Delta_\varphi(P)}_{\Delta_\varphi(Q)}$ is a two-sided ideal, hence we have the following definition.

Definition 2.1 ([6]). The *extended Brauer quotient* associated to the P -interior algebra A and the subgroup K of $\text{Aut}(P)$ is the external direct sum

$$\bar{N}_A^K(P) := \bigoplus_{\varphi \in K} A^{\Delta_\varphi(P)} / \bigoplus_{\varphi \in K} \sum_{Q < P} A^{\Delta_\varphi(P)}_{\Delta_\varphi(Q)} \simeq \bigoplus_{\varphi \in K} A(\Delta_\varphi(P)).$$

Remark 2.2. Note that in this case, one deduces easily from the details given in [6, Section 3] and [7, Section 3] that $\bar{N}_A^K(P)$ is a K -graded algebra, and the map $\text{Br}_P^K := \bigoplus_{\varphi \in K} \text{Br}_{\Delta_\varphi(P)}$ is a homomorphism of K -graded algebras. This fact will become even more transparent in the next section.

2.2. The case of G -interior algebras

In addition to the situation of subsection 2.1 we assume the A is a G -interior algebra, where G is a (not necessarily finite) group, and P is a p -subgroup of G . Conjugation induces the group homomorphisms

$$N_G(P) \rightarrow \text{Aut}(P) \quad \text{and} \quad N_G(P)/C_G(P) \rightarrow \text{Aut}(P), \quad (1)$$

and for the subgroup K in $\text{Aut}(P)$, $N_G^K(P)$ denotes the inverse image of K in $N_G(P)$. If $x \in N_G(P)$, we use denote by φ_x the automorphism of P given by $\varphi_x(u) = u^x = x^{-1}ux$ for all $u \in P$.

In this setting, we obtain some additional properties of the extended Brauer quotient (the details are left to the reader).

Proposition 2.3. *With the above notation, the following statements hold:*

- 1) $\bar{N}_A^K(P)$ is a K -graded $N_G^K(P)$ -interior algebra;
- 2) If $K = N_G(P)/C_G(P)$, then we have the isomorphism

$$\bar{N}_A^K(P) \simeq A(P) \otimes_{kC_G(P)} kN_G(P)$$

of $N_G(P)/C_G(P)$ -graded $N_G(P)$ -interior algebras.

Proof. 1) We only need to notice that any $x \in N_G^K(P)$ verifies $u^{-1}x\phi_x(u) = x$.

2) We define the $N_G(P)/C_G(P)$ -graded map

$$A(P) \otimes_{kC_G(P)} kN_G(P) \rightarrow \tilde{N}_A^K(P), \quad \bar{a} \otimes x \mapsto \overline{ax},$$

whose restriction to the identity component is an isomorphism. \square

Remark 2.4. Note that if $K = N_G(P)/C_G(P)$, then $\tilde{N}_{\mathcal{O}G}^K(P)$ is just the group algebra $kN_G(P)$ considered with the obvious K -grading. Moreover, the construction of $\tilde{N}_A^K(P)$ is clearly functorial in A , so the $N_G(P)$ -interior algebra structure of $\tilde{N}_A^K(P)$ comes from applying the construction to the algebra map $\mathcal{O}G \rightarrow A$.

3. An alternative construction

3.1. The $\mathcal{O}P$ -interior algebra A admits an obvious $(\mathcal{O}P, \mathcal{O}P)$ -bimodule structure. Consider the group algebra $\mathcal{O}[P \rtimes \text{Aut}(P)]$, of the semidirect product $P \rtimes \text{Aut}(P)$. This algebra is also a left $\mathcal{O}P$ -module, hence it makes sense to consider the $\text{Aut}(P)$ -graded $(\mathcal{O}P, \mathcal{O}P)$ -bimodule

$$\tilde{A} := A \otimes_P \mathcal{O}(P \rtimes \text{Aut}(P)).$$

We may also use the isomorphism

$$\tilde{A} \simeq A \otimes_{\mathcal{O}} \mathcal{O} \text{Aut}(P)$$

of \mathcal{O} -modules, which becomes an isomorphism of $(\mathcal{O}P, \mathcal{O}P)$ -bimodules, by defining the bimodule structure of $A \otimes_{\mathcal{O}} \mathcal{O} \text{Aut}(P)$ as follows:

$$u(a \otimes \phi)v = u \cdot a \cdot \phi(v) \otimes \phi,$$

for $u, v \in P$ and $\phi \in \text{Aut}(P)$. Then we regard $A \otimes_{\mathcal{O}} \mathcal{O} \text{Aut}(P)$ as an $\text{Aut}(P)$ -graded P -algebra with P -action given by

$$(a \otimes \phi)^u = u^{-1} \cdot a \cdot \phi(u) \otimes \phi,$$

With the notations of Sections 2 and 3 we have:

Theorem 3.2. *There is an isomorphism*

$$\tilde{A}(P) \simeq \tilde{N}_A^{\text{Aut}(P)}(P)$$

of $\text{Aut}(P)$ -graded algebras, where $\tilde{A}(P)$ is the usual Brauer quotient of \tilde{A} .

Proof. As the p -group P is a normal subgroup of $P \rtimes \text{Aut}(P)$, we get the decomposition

$$\tilde{A}(P) = \bigoplus_{\varphi \in \text{Aut}(P)} (A \otimes (1, \varphi))(P).$$

If $a \otimes (1, \varphi) \in (A \otimes (1, \varphi))^P$, then

$$u^{-1} \cdot a \otimes (1, \varphi) \cdot u = u^{-1} \cdot a \otimes (1, \varphi)(u, 1) = u^{-1} a \varphi(u) \otimes (1, \varphi) = a \otimes (1, \varphi).$$

Then $a \in A^{\Delta_\varphi(P)}$, and consequently

$$(A \otimes (1, \varphi))(P) \rightarrow \bar{N}_A^\varphi(P), \quad \overline{a \otimes (1, \varphi)} \mapsto \bar{a},$$

is a well-defined map of \mathcal{O} -modules for every $\varphi \in \text{Aut}(P)$. We extend this map to a $\text{Aut}(P)$ -graded map between these two modules and we notice that, with all the above identifications, it is actually an isomorphism of algebras. \square

Remark 3.3. We often use subgroups of P , and we obviously have the isomorphism

$$(A \otimes_{\mathcal{O}Q} \mathcal{O}[Q \rtimes K])(Q) \simeq \bar{N}_A^K(Q)$$

of K -graded algebras, for any subgroups $Q \leq P$ and $K \leq \text{Aut}(Q)$.

4. The extended Brauer quotient of a group graded algebra

In this paragraph we set $\bar{G} := G/H$, where H is a normal subgroup of the finite group G , P is a p -subgroup of G , and let

$$A := B \otimes_{\mathcal{O}H} \mathcal{O}G$$

for some H -interior G -algebra B , so A is the G -interior \bar{G} -graded algebra induced from B .

The following lemma says that we restrict ourselves, without loss, to a certain subgroup of $\text{Aut}(P)$.

Lemma 4.1. *Let $\varphi \in \text{Aut}(P)$, and let $O(\bar{x})$ be the orbit of $\bar{x} \in \bar{G}$ under the action of $\Delta_\varphi(P)$ on \bar{G} . If $|O(\bar{x})| \neq 1$ then $(\bigoplus_{\bar{z} \in O(\bar{x})} B \otimes z)(\Delta_\varphi(P)) = 0$.*

Proof. Consider the element $a = \sum b_{z_i} \otimes z_i$ such that $u^{-1} a \varphi(u) = a$. Since the elements z_i are all representatives of the classes of an orbit, we can choose them such that for any $u \in P$ we obtain $u^{-1} z_i \varphi(u) = z_j$. It follows that $b_{z_i}^u = b_{z_j}$, and then there is one element, say b_z , such that $b_z^u = b_{z_i}$ for any i and any $u \in P$. Hence

$$a = \text{Tr}_{\Delta_\varphi(Q)}^{\Delta_\varphi(P)}(b_z \otimes z),$$

where Q is the stabilizer of $b_z \otimes z$ in P . \square

4.2. The above lemma gives the motivation to introduce two subgroups of $\text{Aut}(P)$, because it implies that $(A \otimes \phi)(P) = 0$ for $\phi \in \text{Aut}(P)$ not satisfying $\overline{\phi(u)} = \overline{u^g}$ in \bar{G} , for some $g \in G$. So let

$$\text{Aut}_{\bar{G}}(P) = \{\phi \in \text{Aut}(P) \mid \overline{\phi(u)} = \overline{u^g} \text{ for some } \bar{g} \in \bar{G} \text{ and for any } u \in P\}$$

and

$$\text{Aut}_{\bar{1}}(P) = \{\phi \in \text{Aut}(P) \mid \overline{\phi(u)} = \bar{u} \text{ for any } u \in P\}.$$

Denote also

$$K := \text{Aut}_{\bar{G}}(P), \quad K_1 := \text{Aut}_{\bar{1}}(P),$$

and let $\tilde{A} := A \otimes_{\mathcal{O}P} \mathcal{O}[P \rtimes K]$ as in Section 3.

Finally, let $N_{\bar{G}}^K(\bar{P})$ denote the subgroup of $N_{\bar{G}}(\bar{P})$ whose elements define an element of K and let U be the inverse image of $N_{\bar{G}}^K(\bar{P})$ in G . Also let U' be the inverse image of $C_{\bar{G}}(\bar{P})$ in G . Observe that $N_G^K(P) = N_G(P)$ and $N_G^{K_1}(P) = N_G(P) \cap U'$.

Lemma 4.3. *The group $\text{Aut}_{\bar{1}}(P)$ is a normal subgroup of $\text{Aut}_{\bar{G}}(P)$, hence U' is normal in U . Furthermore, we have the isomorphisms*

$$\text{Aut}_{\bar{G}}(P) / \text{Aut}_{\bar{1}}(P) \simeq N_{\bar{G}}^K(\bar{P}) / C_{\bar{G}}(\bar{P}) \simeq U / U'.$$

Proof. If $\phi_1 \in \text{Aut}_{\bar{1}}(P)$ then $\overline{\phi_1(u)} = \bar{u}$ for all $u \in P$. Hence, if $\phi \in \text{Aut}_{\bar{G}}(P)$ with $\overline{\phi(u)} = \overline{u^g}$, we get

$$\overline{(\phi^{-1} \circ \phi_1 \circ \phi)(u)} = \overline{(\phi_1 \circ \phi)(u)^{g^{-1}}} = \overline{\phi(u)^{g^{-1}}} = \bar{u}.$$

Further if $x \in \bar{g}$ then $\overline{\phi(u)} = \overline{u^g} = \overline{u^x}$ and then $\overline{g^{-1}x} \in C_{\bar{G}}(\bar{P})$. With all of the above, the map

$$\text{Aut}_{\bar{G}}(P) \ni \phi \mapsto \bar{g} \in N_{\bar{G}}(\bar{P})$$

gives the first isomorphism. The second isomorphism is obvious. \square

We will denote by $\tilde{\phi}$ the image of ϕ in the quotient group $\text{Aut}_{\bar{G}}(P) / \text{Aut}_{\bar{1}}(P)$.

Theorem 4.4. *The algebra $\tilde{N}_A^{\text{Aut}(P)}(P)$, as constructed in 2.1, is the U/U' -graded $N_G(P)$ -interior algebra $\tilde{N}_A^K(P)$ with identity component the $N_G^{K_1}(P)$ -interior algebra*

$$\tilde{N}_A^{K_1}(P) = \bigoplus_{g' \in U'/H} \tilde{N}_{B \otimes g'}^{K_1}(P),$$

and for any $\tilde{g} \in U/U'$ (corresponding to $\tilde{\phi}$), the \tilde{g} -component is

$$\bar{N}_A^K(P)_{\tilde{g}} = \bigoplus_{\phi_{\bar{z}} \in \tilde{\phi}; \bar{z} \in \tilde{g}} (B \otimes z)(\Delta_{\phi_{\bar{z}}}(P)),$$

where $\phi_{\bar{z}} \in \tilde{\phi}$ satisfies $\overline{\phi_{\bar{z}}(u)} = \overline{u^z}$, for any $u \in P$.

Proof. By Lemma 4.1, we obtain the following decomposition of the extended Brauer quotient

$$\begin{aligned} \bar{N}_A^{\text{Aut}(P)}(P) &= \left(\bigoplus_{\phi \in K_1} \bar{N}_A^\phi(P) \right) \oplus \bar{N}_A^{K \setminus K_1}(P) \\ &= \left(\bigoplus_{\bar{g}' \in U'/H} \bar{N}_{B \otimes g'}^{K_1}(P) \right) \\ &\quad \oplus \bigoplus_{\tilde{\phi} \in K/K_1} \left(\bigoplus_{\phi_{\bar{z}} \in \tilde{\phi}; \bar{z} \in \tilde{g}} (B \otimes z)(\Delta_{\phi_{\bar{z}}}(P)) \right), \end{aligned}$$

where in the second sum $\tilde{\phi}$ corresponds to \tilde{g} .

We see that, for any \tilde{g} and any $\bar{z} \in \tilde{g}$,

$$B \otimes z = B \otimes zx^{-1}x = (B \otimes zx^{-1}) \cdot (B \otimes x),$$

for any $x \in U'$. Then

$$\bar{N}_A^K(P)_{\tilde{g}} \cdot \bar{N}_A^{K_1}(P) = \bar{N}_A^{K_1}(P) \cdot \bar{N}_A^K(P)_{\tilde{g}} = \bar{N}_A^K(P)_{\tilde{g}}.$$

The fact that this algebra is $N_G(P)$ -interior is immediate since for any $x \in N_G(P)$ the element $1 \otimes x$ is $\Delta_{\phi_{\bar{x}}}(P)$ -invariant. \square

Remark 4.5. 1) The fact that in the above theorem every \tilde{g} -component of $\bar{N}_A^K(P)$ is a direct sum suggests that this algebra actually has a finer grading than stated. Indeed, it is not difficult to see that $\bar{N}_A^{K_1}(P)$ is graded by the group

$$\tilde{K}_1 := \{(\phi, \bar{g}) \mid \phi \in K_1, \bar{g} \in U'/H \text{ such that } \overline{\phi(u)} = \overline{u^g}\},$$

and in general, $\bar{N}_A^K(P)$ is graded by the group

$$\tilde{K} := \{(\phi, \bar{g}) \mid \phi \in K, \bar{g} \in U/H \text{ such that } \overline{\phi(u)} = \overline{u^g}\}.$$

2) Applying the construction to the group algebra $\mathcal{O}G$ yields $\bar{N}_{\mathcal{O}G}^K(P) = kN_G(P)$. The map

$$N_G(P) \rightarrow \tilde{K}, \quad g \mapsto (\phi_g, \bar{g}),$$

where $\phi_g(u) = gug^{-1}$, is a group homomorphism with kernel $C_H(P)$. The \tilde{G} -graded algebra map $\mathcal{O}G \rightarrow A$ induces by functoriality the \tilde{K} -graded algebra map

$$kN_G(P) \rightarrow \bar{N}_A^K(P).$$

3) Observe finally that the construction of $\bar{N}_A^K(P)$ does not require the G -interiority of A . We only need a \tilde{G} -graded algebra A , a group homomorphism $P \rightarrow \tilde{G}$ inducing a \tilde{G} -grading on the group algebra $\mathcal{O}P$, and a homomorphism $\mathcal{O}P \rightarrow A$ of \tilde{G} -graded algebras.

4.6. Next, we establish the connection between $\bar{N}_A^K(P) \simeq \tilde{A}(P)$ and the extended Brauer quotient $\bar{N}_B^{K_1}(P)$ of the H -interior G -algebra B , introduced in [3]. Recall that $\bar{N}_B^{K_1}(P)$ is an $N_H^{K_1}(P)$ -interior $N_G(P)$ -algebra constructed formally as in Section 1 above. One can easily see from the definition in [3, Section 2] that $\bar{N}_B^{K_1}(P)$ is actually a K_1 -graded $N_G^{K_1}(P)$ -interior $N_G(P)$ -algebra.

Let $Q := P \cap H$. Then, as in Section 3, let

$$\tilde{B} := B \otimes_{\mathcal{O}Q} \mathcal{O}(Q \rtimes K_1) \simeq B \otimes_{\mathcal{O}} \mathcal{O}K_1.$$

Proposition 4.7. *The \mathcal{O} -module \tilde{B} is a $\mathcal{O}\Delta(P \times P)$ -module via*

$$(b \otimes (1, \varphi))^{(u, u)} = b^u \otimes u^{-1} \cdot (1, \varphi) \cdot u = b^u \otimes (u^{-1} \varphi(u), \varphi) = b^u u^{-1} \varphi(u) \otimes (1, \varphi),$$

for any $u \in P$, $b \in B$ and $\varphi \in K_1$. Furthermore, we have the isomorphism

$$\tilde{B}(\Delta(P \times P)) \simeq \bar{N}_B^{K_1}(P)$$

of K_1 -graded $N_G^{K_1}(P)$ -interior $N_G(P)$ -algebras with identity component $B(P)$.

Proof. It is clear that for any $\varphi \in K_1$ we have $\varphi(u) \in Q$ for any $u \in Q$, hence K_1 acts on Q and \tilde{B} is a well-defined $\Delta(P \times P)$ -module and we have

$$\tilde{B}(\Delta(P \times P)) = \bigoplus_{\varphi \in K_1} (B \otimes (1, \varphi))(\Delta(P \times P)).$$

For any $\varphi \in K_1$ the map

$$(B \otimes (1, \varphi))(\Delta(P \times P)) \ni \overline{b \otimes (1, \varphi)} \mapsto \bar{b} \in \bar{N}_B^\varphi(P)$$

is an isomorphism of k -vector spaces. The direct sum of these maps is the required algebra isomorphism. \square

Remark 4.8. 1) According to Theorem 3.2 and Theorem 4.4, we have the decompositions

$$\begin{aligned}\tilde{A}(P) &\simeq \tilde{N}_A^K(P) = \tilde{N}_{B \otimes_H U'}^{K_1}(P) \oplus \tilde{N}_A^{K \setminus K_1}(P) \\ &= \tilde{N}_B^{K_1}(P) \oplus \left(\bigoplus_{\bar{x} \in U'/H, x \notin H} \tilde{N}_{B \otimes x}^K(P) \right) \oplus \tilde{N}_A^{K \setminus K_1}(P).\end{aligned}$$

The above statements show that the $N_G^{K_1}(P)$ -interior algebra $\tilde{B}(P)$ can be identified with a unitary subalgebra of $\tilde{A}(P)$, and even of $\tilde{N}_A^{K_1}(P)$, such that the $N_G(P)$ -action and the K_1 -grading are preserved. For the particular case of the H -interior G -invariant group algebra $B = \mathcal{O}H$, the component $\tilde{N}_{B \otimes_H U'}^{K_1}(P)$ is the $N_G(P)$ -algebra studied in [2, Section 5].

2) The Brauer quotient $B(P)$ of B is a $C_H(P)$ -interior $N_G(P)$ -algebra. The argument of Proposition 2.3 implies that the induced algebra

$$B(P) \otimes_{kC_H(P)} kN_G(P)$$

is isomorphic to a \tilde{K} -graded subalgebra of $\tilde{N}_A^K(P)$, while

$$B(P) \otimes_{kC_H(P)} kC_G(P)$$

is isomorphic to a \tilde{K}_1 -graded subalgebra of $\tilde{N}_B^K(P)$.

5. Tensor products of algebras

Recall that if A and A' are two G -graded algebras, then the diagonal subalgebra of the $G \times G$ -graded algebra $A \otimes A'$ is the G -graded subalgebra

$$\Delta(A \otimes A') := \bigoplus_{g \in G} (A_g \otimes A'_g).$$

The following result is an extension of [6, Proposition 3.9]

Theorem 5.1. *Assume that A and A' are two G -interior algebras such that A' has a $P \times P$ -invariant \mathcal{O} -basis, and let K be a subgroup of $\text{Aut}(P)$.*

1) *There is an isomorphism*

$$\tilde{N}_{A \otimes_{\mathcal{O}} A'}^K(P) \simeq \Delta(\tilde{N}_A^K(P) \otimes_k \tilde{N}_{A'}^K(P))$$

of K -graded $N_G^K(P)$ -interior algebras.

2) Assume in addition that, as K -graded $N_G^K(P)$ -interior algebras,

$$\bar{N}_A^K(P) \simeq A(P) \otimes_k kK.$$

Then

$$\bar{N}_{A \otimes_{\mathcal{O}} A'}^K(P) \simeq A(P) \otimes_k \bar{N}_{A'}^K(P)$$

as K -graded $N_G(P)$ -interior algebras.

Proof. 1) We consider the $K \times K$ -graded $N_G^K(P)$ -interior algebra

$$\bar{N}_A^K(P) \otimes_k \bar{N}_{A'}^K(P) = \bigoplus_{\varphi, \psi \in K} \bar{N}_A^\varphi(P) \otimes_k \bar{N}_{A'}^\psi(P),$$

whose diagonal subalgebra

$$\Delta(\bar{N}_A^K(P) \otimes_k \bar{N}_{A'}^K(P)) = \bigoplus_{\varphi \in K} \bar{N}_A^\varphi(P) \otimes_k \bar{N}_{A'}^\varphi(P)$$

is an $N_G^K(P)$ -interior K -graded algebra. Due to the inclusion

$$A^{\Delta_\varphi(P)} \otimes_{\mathcal{O}} (A')^{\Delta_\varphi(P)} \subseteq (A \otimes_{\mathcal{O}} A')^{\Delta_\varphi(P)},$$

we obtain an \mathcal{O} -module map

$$A^{\Delta_\varphi(P)} \otimes_{\mathcal{O}} (A')^{\Delta_\varphi(P)} \rightarrow \bar{N}_{A \otimes_{\mathcal{O}} A'}^\varphi(P)$$

sending $a \otimes a'$ to $\overline{a \otimes a'}$. If $c \in A^{\Delta_\varphi(Q)}$ and $c' \in (A')^{\Delta_\varphi(R)}$, for some subgroups Q and R of P then

$$\begin{aligned} \mathrm{Tr}_{\Delta_\varphi(Q)}^{\Delta_\varphi(P)}(c) \otimes \mathrm{Tr}_{\Delta_\varphi(R)}^{\Delta_\varphi(P)}(c') &= \mathrm{Tr}_{\Delta_\varphi(Q)}^{\Delta_\varphi(P)} \left(c \otimes \mathrm{Tr}_{\Delta_\varphi(R)}^{\Delta_\varphi(P)}(c') \right) \\ &= \mathrm{Tr}_{\Delta_\varphi(R)}^{\Delta_\varphi(P)} \left(\mathrm{Tr}_{\Delta_\varphi(Q)}^{\Delta_\varphi(P)}(c) \otimes c' \right) \\ &\in (A \otimes_{\mathcal{O}} A')_{\Delta_\varphi(R)}^{\Delta_\varphi(P)} \cap (A \otimes_{\mathcal{O}} A')_{\Delta_\varphi(Q)}^{\Delta_\varphi(P)}. \end{aligned}$$

This determines an \mathcal{O} -module homomorphism

$$\bar{N}_A^\varphi(P) \otimes \bar{N}_{A'}^\varphi(P) \rightarrow \bar{N}_{A \otimes_{\mathcal{O}} A'}^\varphi(P), \quad \bar{a} \otimes \bar{a'} \mapsto \overline{a \otimes a'}$$

for every $\varphi \in K$. The direct sum of all these homomorphism is a K -graded algebra homomorphism between $\Delta(\bar{N}_A^K(P) \otimes_k \bar{N}_{A'}^K(P))$ and $\bar{N}_{A \otimes_{\mathcal{O}} A'}^K(P)$, which is in fact an isomorphism of interior $N_G^K(P)$ -algebras since by our assumptions we have

$$(A \otimes_{\mathcal{O}} A')(P) \simeq A(P) \otimes_k A'(P).$$

2) By the additional assumption we obtain

$$\Delta(\bar{N}_A^K(P) \otimes_k \bar{N}_{A'}^K(P)) = \bigoplus_{\varphi \in K} (A(P) \otimes_k k\varphi) \otimes \bar{N}_{A'}^{\varphi}(P).$$

We define the k -linear map

$$A(P) \otimes_k \bar{N}_{A'}^{\varphi}(P) \rightarrow (A(P) \otimes_k k\varphi) \otimes \bar{N}_{A'}^{\varphi}(P), \quad \bar{a} \otimes \bar{a}' \mapsto (\bar{a} \otimes \varphi) \otimes \bar{a}',$$

for every $\varphi \in K$. The sum of these maps determine the required isomorphism of K -graded interior $N_G(P)$ -algebras between $A(P) \otimes_k \bar{N}_{A'}^K(P)$ and $\Delta(\bar{N}_A^K(P) \otimes_k \bar{N}_{A'}^K(P))$. \square

Remark 5.2. 1) Statement 2) of the previous theorem applies in the situation of [6, Proposition 3.8]. More precisely, let

$$A = \text{End}_{\mathcal{O}}(N)$$

for an indecomposable endopermutation $\mathcal{O}P$ -module N , such that $A(P) \neq 0$. Let $Q \leq P$, and let δ be the unique local point of Q on A . Let $K := F_A(Q_{\delta})$. Then [6, Proposition 3.8] says that there is an isomorphism

$$\bar{N}_A^K(Q) \simeq A(Q) \otimes_k kK$$

of $N_P^K(Q)$ -interior K -graded algebras.

2) Assume in addition that A' is \bar{G} -graded G -interior as in Section 4, and has a $P \times P$ -invariant \mathcal{O} -basis consisting of \bar{G} -homogeneous elements. Then, by Remark 4.5, the isomorphism in Theorem 5.1. 2) is in fact an isomorphism of \tilde{K} -graded $N_G(P)$ -interior algebras.

6. A correspondence for covering points

In this section we establish a correspondence between covering points in the case of a G -interior algebra that has a stable basis.

6.1. We keep the notations of Section 4, and we assume that the G -interior \tilde{G} -graded algebra $A := B \otimes_{\mathcal{O}H} \mathcal{O}G$ has a $G \times G$ -stable \mathcal{O} -basis consisting of \tilde{G} -homogeneous elements. Further, we assume that A is projective regarded as a left and as a right $\mathcal{O}G$ -module. By these assumptions, B is an H -interior permutation G -algebra, and it is projective both as a left and a right $\mathcal{O}H$ -module.

6.2. We fix a normal subgroup N of G that contains P and a point β of N on B with defect group P . Then our assumptions and [3, Theorem 3.1] imply that $\tilde{\beta} := \text{Br}_P(\beta)$ is a point of $N_N(P)$ on $\tilde{B}(P)$ with defect group P .

We adopt here the definition of covering points from [1]. We say that the point α of G on A covers β if α has defect group P and for any $i \in \alpha$ there is an idempotent $j_1 \in A^N$ that lies in the conjugacy class of β and there is a primitive idempotent $f \in A^N$ belonging to a point with defect group P such that $j_1 f = f j_1 = f$ and $if = fi = f$.

Clearly in this case we consider a particular setting in which a defect group of the points that are covered coincides with a defect group of the points that cover the given ones.

Now we can state our result.

Theorem 6.3. *The Brauer homomorphism*

$$\text{Br}_P : A^P \rightarrow A(P)$$

determines a bijective correspondence between the points of A^G with defect group P that cover β and the points of $\tilde{A}(P)^{N_G(P)}$ with defect group P that cover $\tilde{\beta}$.

Proof. Theorem 3.2 and [6, Proposition 3.3] already provide a bijection between the points of G on A and the points of $N_G(P)$ on $\tilde{A}(P)$ with the same defect group P . Even more, this bijection coincides with the bijection determined by the epimorphism given by the restriction of the Brauer homomorphisms

$$\text{Br}_P : A_P^G \rightarrow A(P)_P^{N_G(P)}.$$

Since N is normal in G , hence $N_N(P)$ is also normal in $N_G(P)$, the fact that this bijection restricts to a bijection between the points that cover β and $\tilde{\beta}$ is an easy verification given by [1, Theorem 3.5]. \square

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